Approximations of wave propagation in one-dimensional multiple scattering problems with random characteristics

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Introduction

Randomness or disorder in the position or properties of a large field of scattering sources has a significant effect on the propagation of waves. A series of recent studies have shown the effect for surface waves travelling through arrays of floating bodies (e.g. Bennetts et al., 2010; Bennetts, 2011; Peter & Meylan, 2009a,b). In those investigations the primary interest was to extract the attenuation rate of waves as a function of the incident wave field and the properties of the field of scatterers. The attenuation is related to localization theory, from which it follows that a wave will decay at the same rate in each individual realisation of the field as the number of scatterers tends to infinity. For computational purposes, in which only a finite number of scatterers is feasible, the attenuation rate fluctuates around the infinite limit and this value is approximated by taking the average of a sufficiently large number of randomly generated simulations, i.e. a Monte-Carlo algorithm (MCA). The change in phase can also be approximated using a MCA. The phase change and attenuation rate provide the real and imaginary components of the effective wavenumber of the multiple-scattering medium, which is the quantity that will be sought in this investigation.

But, when the problem involves a large number of parameters and the randomness is multi-dimensional, it is expensive to use a MCA to determine the functional dependence of the effective wavenumber. The goal is therefore to find or approximate the effective wavenumber without solving the full multiple-scattering problem. Ultimately, the intention is to find a method suitable for the two-dimensional scattering problems that motivate this study. However, the first attempts to identify candidate approaches, which are described in the following sections, are honed on simpler problems in which the multiple scattering is one-dimensional.

Two different problems are examined here with solution methods corresponding to the type of randomness involved. Both problems incorporate the usual assumptions of linear theory and time-harmonic motions. The first problem involves a one-dimensional array of scatterers, where the potential function satisfies Helmholtz equation between the scatterers. Randomness is parameterised in terms of deviations away from an underlying periodic arrangement of the scatterers, for which a dispersion relation is known. Two approaches are considered for this problem. Both are based on isolating an individual cell in the system that contains a single scatterer. The second problem is set in a two-dimensional fluid domain, which has its surface covered by an elastic plate. Therefore, the potential function satisfies Laplace's equation in the fluid domain and a high-order boundary condition at the interface with the elastic plate. Scattering is produced by small variations in the properties of the plate (its mass and rigidity). A multi-scale expansion for the potential is sought, and from the low-order scales an equation that describes the modulation of the amplitude of the wave field, i.e. the envelope, is derived. A small selection of numerical results are shown for the first problem. Further results will be given at the workshop, including results for the second problem.

Transfer matrix methods for positional disorder in an array of point scatterers

Consider an infinite one-dimensional array of point scatterers of strength M, and with positions that are distributed randomly about a periodic arrangement $x_n = nd$ $(n \in \mathbb{Z})$. Let the deviation of scatterer n from its periodic location be $\epsilon_n d$. This is identical to the problem considered by Maurel et al. (2010). Those authors derived two different dispersion relations for the effective wavenumber corresponding to two different approximation methods. The derivation is based on small perturbations of the array away from the periodic structure. Unsurprisingly then, the dispersion relations contain the wavenumber of the periodic medium. Similar approximations will be derived here but without the restriction to small perturbations. Instead, the deviations ϵ_n are chosen randomly from the interval $(-\epsilon/2, \epsilon/2)$, for a chosen $\epsilon \in [0,1]$.

The starting point for both approximations is to note that between the scatterers the potential, ϕ, has the form

$$\phi = a_n \mathrm{e}^{\mathrm{i}k(x-x_n-d/2)} + b_n \mathrm{e}^{-\mathrm{i}k(x-x_n-d/2)} \qquad (x_{n-1}+\epsilon_{n-1} < x < x_n + \epsilon_n),$$

where k is the wavenumber between the scatterers, and the amplitudes a_n and b_n are functions of the deviations ϵ_n $(n \in \mathbb{Z})$. The amplitudes on either side of a scatterer can be related to one another using a transfer matrix, P say, and this relationship is expressed as

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = P_n \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad \text{where} \quad P_n = P(\epsilon_n) = \begin{pmatrix} P_{11}\mathrm{e}^{\mathrm{i}kd} & P_{12}\mathrm{e}^{-2\mathrm{i}kd\epsilon_n} \\ P_{21}\mathrm{e}^{2\mathrm{i}kd\epsilon_n} & P_{22}\mathrm{e}^{-\mathrm{i}kd} \end{pmatrix}. \tag{1}$$

The quantities P_{ij} depend only on the properties of an individual scatterer, i.e. its strength M.

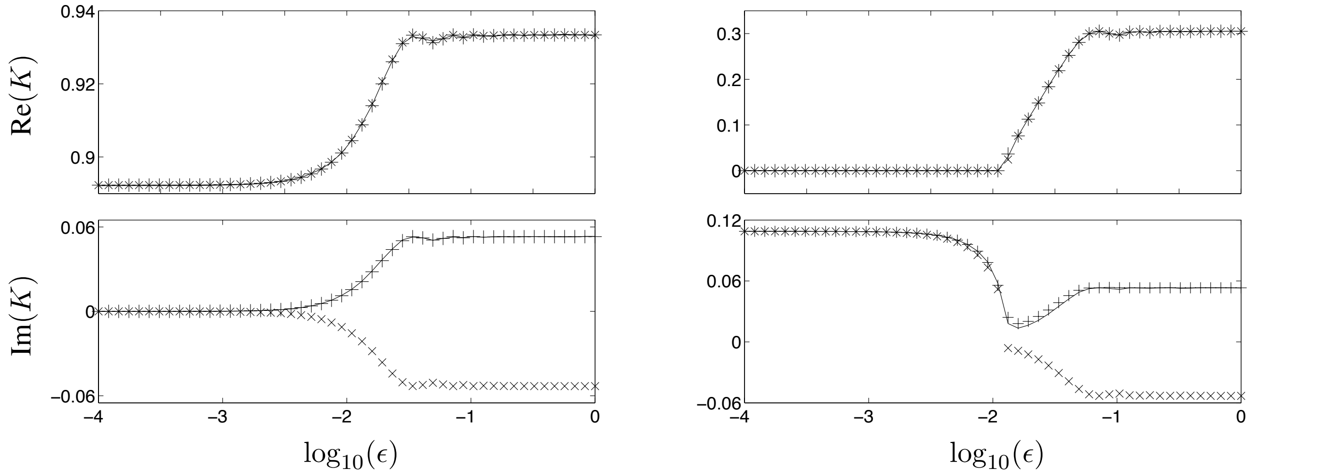

Figure 1: The effective wavenumber as a function of the deviation limit, calculated using the CLA ($\times$), the CPA ($+$) and a MCA ($-$). The underlying periodic medium is in a pass band in the left-hand panels ($kd = 28.4\pi$, $M = 0.67/2\mathrm{i}$), and a stop gap in the right-hand panels ($kd = 14.2\pi$, $M = 0.67/2\mathrm{i}$).

The first approximation results from a closure assumption (CLA). Denote the ensemble average of a quantity with respect to all deviations by angled brackets $\langle\cdot\rangle$, the conditional probability with respect to deviation ϵ_n, i.e. all deviations except ϵ_n, by an additional bracketed subscript $\langle\cdot\rangle_{(\epsilon_n)}$, and the average with respect to only deviation ϵ_n with an unbracketed subscript $\langle\cdot\rangle_{\epsilon_n}$. Taking the full ensemble average of the first component of equation (1) results in

$$\left\langle \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} \right\rangle = \left\langle P_n \left\langle \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\rangle_{(\epsilon_n)} \right\rangle_{\epsilon_n} \approx \langle P_n \rangle_{\epsilon_n} \left\langle \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\rangle,$$

in which the approximation results from the closure assumption. Now, applying the ansatz $\langle a_n \rangle = a\mathrm{e}^{\mathrm{i}Kn}$ and $\langle b_n \rangle = b\mathrm{e}^{\mathrm{i}Kn}$ in the approximation it is simple to show that the term $\mathrm{e}^{\mathrm{i}K}$ is an eigenvalue of the average transfer matrix $\langle P_n \rangle_{\epsilon_n}$, and the amplitudes a and b are the entries of the corresponding eigenvector.

The second method is a form of the coherent potential approximation (CPA). The CPA reduces the problem to the consideration of a single cell, containing one scatterer, in the random media. Translating this idea to the transfer matrix approach leads to the expression

$$\mathrm{e}^{\mathrm{i}K} T \mathbf{v}_+ = P(\hat{\epsilon})\big(\mathbf{v}_+ + R\mathbf{v}_-\big) \quad (\epsilon/2 < \hat{\epsilon} < \epsilon/2), \tag{2}$$

The vectors $\mathbf{v}_\pm$ relate to the rightward (+) and leftward (-) travelling waves that propagate through the random media, and, along with the effective wavenumber, are functions of the limit ϵ only. The quantities R and T are, respectively, reflection and transmission coefficients for the cell, and are therefore functions of the deviation $\hat{\epsilon}$.

Expression (2) contains two equations in the four unknowns R, T, K and $\hat{v}$, where $\mathbf{v}_+ = (1, \hat{v})^T$. (By symmetry $\mathbf{v}_- = (\hat{v}, 1)^T$.) However, it is useful to consider $\hat{T} = \mathrm{e}^{\mathrm{i}K}$ as a dummy unknown replacing T and K. This new unknown can be eliminated from the transfer matrix relationship and further manipulations then give the reflection coefficient as a function of $\hat{v}$, which will be denoted $R = f(\hat{v})$. The CPA is based on the assumption that the cell behaves exactly as the random media when averaged, in other words: (i) $\langle R \rangle_{\hat{\epsilon}} = 0$, and (ii) $\langle \hat{T} \rangle_{\hat{\epsilon}} = \mathrm{e}^{\mathrm{i}K}$. Substituting $R = f(\hat{v})$ into (i) and solving for $\hat{v}$ (using a nonlinear minimisation routine) gives R. The dummy variable $\hat{T}$ may subsequently be obtained via the transfer matrix relation, and it is separated into the original variables T and K using condition (ii).

Two examples of the effective wavenumbers produced by the approximations, as functions of the deviation limit, ϵ, are shown in figure 1. The panels on the left are for parameters in which the underlying periodic medium is in a pass band, and on the right is in a stop gap. (The latter is a problem for which numerical results were given in Maurel et al., 2010.) Results produced by a Monte Carlo algorithm are also provided for reference. Note that, for the pass band and the stop gap, when ϵ is greater than approximately 0.1 the imaginary part of K is constant. This value is the natural logarithm of $|P_{22}|$.

The CPA gives accurate predictions for all cases considered. However, the results given by the CLA are more complicated. Although the real part of the effective wavenumber is captured by the CLA, the imaginary part is in many cases negative rather than positive. (Typically the eigenvalues of $\langle P_n \rangle_{\epsilon_n}$ gives effective wavenumbers that are symmetric with respect to the real axis.) The CLA employed here is first order, as opposed to the second-order closure assumption (a Quasi-crystalline approximation, QCA) used by Maurel et al. (2010). There is an interesting link between the CLA and the QCA. They both give approximations that are equivalent to solving the period problem for scatterers that have reflections and transmission coefficients that are the average of those of a single cell. The CLA takes these values directly from the cell, whereas the QCA takes the values from only one side of the cell. Some aspects of the CLA can be deduced by noting that one of the entries of the transfer matrix is the reciprocal of the transmission coefficient.

Envelope equation for a rough floating elastic beam

As a second problem, we consider two-dimensional water-wave attenuation of time-harmonic waves by an infinitely long floating rough elastic beam of shallow draft. We use a two-scale approach similar to that of Mei et al. (2005, §7.4) and Mei & Hancock (2003), who considered ocean-wave scattering by a random sea bed, and we also refer to the references therein for relations to other areas of wave scattering. Thus, we derive an equation for the average envelope of the random wave fields, which is related to the attenuation coefficient characterising the decay of wave energy over long distances.

The two length scales under consideration are the small scale ℓ, which is of the order of the wave length $2\pi/\kappa$ and the random perturbations and has a coordinate denoted by x, and the observation scale L, over which attenuation is observed, whose coordinate is denoted by x_2. It is assumed that the scales are related by $L = \varepsilon^2 \ell$ for a small $\varepsilon \ll 1$, which is the same scaling considered by Mei et al. (2005, §7.4). In these terms, the beam stiffness is $b + \varepsilon\beta(x)$ and its mass is $g + \varepsilon\gamma(x)$, where b and g are constants and $\beta \sim O(1) \sim \kappa\ell$ and $\gamma \sim O(1)$ are real random functions distributed with zero mean.

We heuristically adopt a multi-scale expansion for the complex time-harmonic potential ϕ

$$\phi(x) = \phi_0(x, x_2) + \varepsilon\phi_1(x, x_2) + \varepsilon^2\phi_2(x, x_2) + \dots,$$

where $x_2 = \varepsilon^2 x$, and we determine the equations to be satisfied by each of the ϕ_j in what follows. Note that $\partial_x \phi = \sum_j \varepsilon^j(\partial_x\phi_j + \varepsilon^2\partial_{x_2}\phi_j)$ by an obvious application of the chain rule.

In what follows, we omit stating the radiation conditions for brevity. Writing α for the quotient of the square of the radian frequency and gravitational acceleration and h for the constant water depth, on the scale of order ε^0 we obtain

$$\begin{aligned}
\nabla_x^2\phi_0 &= 0, && x \in (-\infty,\infty),\ z \in (-h,0),\\
\partial_z\phi_0 &= 0, && x \in (-\infty,\infty),\ z = -h,\\
\mathcal{L}\phi_0 := (b\partial_x^4 - \alpha g + 1)\partial_z\phi_0 - \alpha\phi_0 &= 0, && x \in (-\infty,\infty),\ z = 0.
\end{aligned}$$

This system of equations does not involve any random functions and is thus entirely deterministic. Restricting ourselves to a wave travelling from left to right, we have

$$\phi_0 = \langle\phi_0\rangle = Af(z)\mathrm{e}^{\mathrm{i}\kappa x},$$

where $f(z) = \cosh\kappa(z+h)/\cosh\kappa h$, $A = A(x_2)$ denotes the leading-order wave amplitude and the wavenumber κ satisfies the usual dispersion relation of an elastic beam of constant stiffness b and mass g, i.e.

$$\kappa\tanh\kappa h = \frac{\alpha}{b\kappa^4 - \alpha g + 1}. \tag{4}$$

On the next scale, ε^1, we find

$$\begin{aligned}
\nabla_x^2\phi_1 &= 0, && x \in (-\infty,\infty),\ z \in (-h,0), && \text{(5a)}\\
\partial_z\phi_1 &= 0, && x \in (-\infty,\infty),\ z = -h, && \text{(5b)}\\
\mathcal{L}\phi_1 &= -(\partial_x^2(\beta\partial_x^2) - \alpha\gamma)\partial_z\phi_0, && x \in (-\infty,\infty),\ z = 0. && \text{(5c)}
\end{aligned}$$

Obviously, $\langle\phi_1\rangle = 0$ (yet this fact is not used in what follows). Making use of the corresponding Green's function $G(|x - x'|, z)$, which satisfies (5) but with the right-hand side of (5c) replaced by $\delta(x - x')$ (see Fox & Chung, 1998 and Evans & Porter, 2003 for details), the solution can be written as

$$\begin{aligned}
\phi_1(x,z) &= -\int_{-\infty}^{\infty} \left[(\partial_{x'}^2(\beta\partial_{x'}^2) - \alpha\gamma)\partial_z\phi_0\right] G(|x - x'|, z)\,\mathrm{d}x'\\
&= -A(\partial_z f)\int_{-\infty}^{\infty} \left[(\partial_{x'}^2(\beta\partial_{x'}^2) - \alpha\gamma)\mathrm{e}^{\mathrm{i}\kappa x'}\right] G(|x - x'|, z)\,\mathrm{d}x'. && \text{(6)}
\end{aligned}$$

On the ε^2-scale, we find

$$\begin{aligned}
&\nabla_x^2 \phi_2 = -2\partial_{x_2}\partial_x \phi_0, && x \in (-\infty,\infty),\ z \in (-h,0),\\
&\partial_z \phi_2 = 0, && x \in (-\infty,\infty),\ z = -h,\\
&\mathcal{L}\phi_2 = -(\partial_x^2(\beta\partial_x^2) - \alpha\gamma)\partial_z\phi_1, && x \in (-\infty,\infty),\ z = 0.
\end{aligned}$$

To solve for $\langle \phi_2 \rangle$, we let $\langle \phi_2 \rangle = \mathrm{e}^{\mathrm{i}\kappa x} F(x_2, z)$, so that F has to satisfy

$$\partial_z^2 F - \kappa^2 F = -2\mathrm{i}\kappa f(\partial_{x_2} A), \qquad z \in (-h,0), \tag{8a}$$

$$\partial_z F = 0, \qquad z = -h, \tag{8b}$$

and

$$(b\kappa^4 - \alpha g + 1)\partial_z F - \alpha F = A\mathrm{e}^{-\mathrm{i}\kappa x}(\partial_z^2 f) \left\langle \left(\partial_x^2(\beta\partial_x^2) - \alpha\gamma\right) \int_{-\infty}^{\infty} \left[(\partial_{x'}^2(\beta\partial_{x'}^2) - \alpha\gamma)\mathrm{e}^{\mathrm{i}\kappa x'}\right] G \,\mathrm{d}x' \right\rangle, \tag{8c}$$

at $z = 0$ by virtue of (6). The right-hand side of the surface condition (8c) can be rewritten as

$$A\kappa^2 \left\langle \mathrm{e}^{-\mathrm{i}\kappa x} \left(\partial_x^2(\beta\partial_x^2) - \alpha\gamma\right) \int_{-\infty}^{\infty} \left(\kappa^2(\kappa^2\beta - 2\mathrm{i}\kappa(\partial_{x'}\beta) - (\partial_{x'}^2\beta)) + \alpha\gamma\right)\mathrm{e}^{\mathrm{i}\kappa x'} G(|x - x'|, 0)\,\mathrm{d}x' \right\rangle =: A\kappa^2\zeta. \tag{9}$$

Under the assumption of stationarity and the existence of correlation functions ρ_i such that

$$\langle \beta(x)\beta(x')\rangle = \sigma_1^2\rho_1(\xi), \quad \langle \gamma(x)\gamma(x')\rangle = \sigma_2^2\rho_2(\xi), \quad \langle \beta(x)\gamma(x')\rangle = \sigma_3^2\rho_3(\xi),$$

where $\xi = |x - x'|$ and the root-mean-square amplitudes of the disorder, σ_i, are either constant or dependent on x_2 only, the quantity ζ can be shown to be a complex constant. (Note that such an assumption is reasonable even for the mixed term if the randomness of the stiffness and the mass originate from a random beam thickness e.g.)

Application of Green's second identity to f and to F gives

$$\int_{-h}^{0} \{\kappa^2 f F - f\left(\kappa^2 F - 2\mathrm{i}\kappa f(\partial_{x_2} A)\right)\}\,\mathrm{d}z = F(x_2, 0)\kappa \tanh \kappa h - \partial_z F(x_2, 0).$$

Rearrangement and use of the dispersion relation (4) as well as (8c) in combination with (9) leaves

$$2\mathrm{i}(\kappa\partial_{x_2} A) \int_{-h}^{0} f^2(z)\,\mathrm{d}z = F(x_2, 0)\frac{\alpha}{b\kappa^4 - \alpha g + 1} - \partial_z F(x_2, 0) = \frac{-A\kappa^2\zeta}{b\kappa^4 - \alpha g + 1},$$

which, after some algebra, leads to the final result

$$\alpha\left(1 + \frac{2\kappa h}{\sinh 2\kappa h}\right)\partial_{x_2} A = \mathrm{i}\kappa^3\zeta A,$$

which is an equation for the envelope A.

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